BOUNDED-DEGREE GRAPHS HAVE
ARBITRARILY LARGE QUEUE-NUMBER

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ABSTRACT. It is proved that there exist graphs of bounded degree with arbitrarily large queue-number. In particular, for all $\Delta \geq 3$ and for all sufficiently large n, there is a simple Δ-regular n-vertex graph with queue-number at least $c\sqrt{\Delta}n^{1/2-1/\Delta}$ for some absolute constant c.

1. INTRODUCTION

We consider graphs possibly with loops but with no parallel edges. A graph without loops is *simple*. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. If $S \subseteq E(G)$ then $G[S]$ denotes the spanning subgraph of G with edge set S. We say G is *ordered* if $V(G) = \{1, 2, \ldots, |V(G)|\}$. Let G be an ordered graph. Let $\ell(e)$ and $r(e)$ denote the endpoints of each edge $e \in E(G)$ such that $\ell(e) \leq r(e)$. Two edges e and f are *nested* and f is *nested inside* e if $\ell(e) < \ell(f)$ and $r(f) < r(e)$. An ordered graph is a *queue* if no two edges are nested. Observe that the left and right endpoints of the edges in a queue are in first-in-first-out order—hence the name 'queue'. An ordered graph G is a *k-queue* if there is a partition $\{E_1, E_2, \ldots, E_k\}$ of $E(G)$ such that each $G[E_i]$ is a queue.

Let G be an (unordered) graph. A *k-queue layout* of G is a k-queue that is isomorphic to G. The *queue-number* of G is the minimum integer k such that G has a k-queue layout. Queue layouts and queue-number were introduced by Heath et al. [11, 12] in 1992, and have applications in sorting permutations [9, 13, 18, 20, 24], parallel process scheduling [3], matrix computations [19], and graph drawing [4, 5]. Other aspects of queue layouts have been studied in [6, 7, 8, 10, 21, 22, 25].

Prior to this work it was unknown whether graphs of bounded degree have bounded queue-number. The main contribution of this note is to prove that there exist graphs of bounded degree with arbitrarily large queue-number.

Theorem 1. *For all $\Delta \geq 3$ and for all sufficiently large $n > n(\Delta)$, there is a simple Δ-regular n-vertex graph with queue-number at least $c\sqrt{\Delta}n^{1/2-1/\Delta}$ for some absolute constant c.*

The best known upper bound on the queue-number of a graph with maximum degree Δ is $\mathsf{e}(\Delta n/2)^{1/2}$ due to Dujmović and Wood [7] (where e is the base of the natural logarithm). Observe that for large Δ, the lower bound in Theorem 1 tends toward this

upper bound. Although for specific values of Δ a gap remains. For example, for $\Delta = 3$ we have an existential lower bound of $\Omega(n^{1/6})$ and a universal upper bound of $\mathcal{O}(n^{1/2})$.

2. Proof of Theorem 1

The proof of Theorem 1 is modelled on a similar proof by Barát et al. [1]. Basically, we show that there are more graphs Δ-regular graphs than graphs with bounded queue-number. The following lower bound on the number of Δ-regular graphs is a corollary of more precise bounds due to Bender and Canfield [2], Wormald [26], and McKay [17]; see [1].

Lemma 1 ([2, 17, 26]). *For all integers $\Delta \geq 1$ and for sufficiently large $n \geq n(\Delta)$, the number of labelled simple Δ-regular n-vertex graphs is at least*

$$\left(\frac{n}{3\Delta}\right)^{\Delta n/2}$$

It remains to count the graphs with bounded queue-number. We will need the following two lemmas from the literature, whose proofs we include for completeness. A *rainbow* in an ordered graph is a set of pairwise nested edges.

Lemma 2 ([7, 12]). *An ordered graph G is a k-queue if and only if G has no $(k+1)$-edge rainbow.*

Proof. The necessity is obvious. For the sufficiency, suppose G has no $(k + 1)$-edge rainbow. For every edge e of G, if $i - 1$ edges are pairwise nested inside e, then assign e to the i-th queue. □

Lemma 3 ([7]). *Every ordered n-vertex graph with no two nested edges has at most $2n - 1$ edges.*

Proof. If $v + w = x + y$ for two distinct edges vw and xy, then vw and xy are nested. The result follows since $2 \leq v + w \leq 2n$. □

Let $g(n)$ be the number of queues on n vertices. To bound $g(n)$ we adapt a proof of a more general result by Klazar [15]; also see [16, 23] for other related and more general results.

Lemma 4. $g(n) \leq c^n$ *for some absolute constant c.*

Proof. Say G is an ordered n-vertex graph. Let G' be an ordered $2n$-vertex graph obtained by the following *doubling* operation. For every edge vw of G, add to G' a nonempty set of edges between $\{2v - 1, 2v\}$ and $\{2w - 1, 2w\}$, no pair of which are nested.

Every ordered $2n$-vertex ordered graph with no two nested edges can be obtained from some ordered n-vertex graph with no two nested pair of edges by a *doubling* operation. To see this, merge every second pair of vertices, introduce a loop between merged vertices, and replace any resulting parallel edges by a single edge. The ordered graph that is obtained has no nested pair of edges.

In the doubling operation, there are 11 possible ways to add a nonempty set of non-nested edges between $\{2v - 1, 2v\}$ and $\{2w - 1, 2w\}$, as illustrated in Figure 1. Thus $g(2n) \leq 11^{2n-1} \cdot g(n)$, since G' has at most $2n - 1$ edges by Lemma 3. It follows that $g(n) \leq 11^{2n}$. □
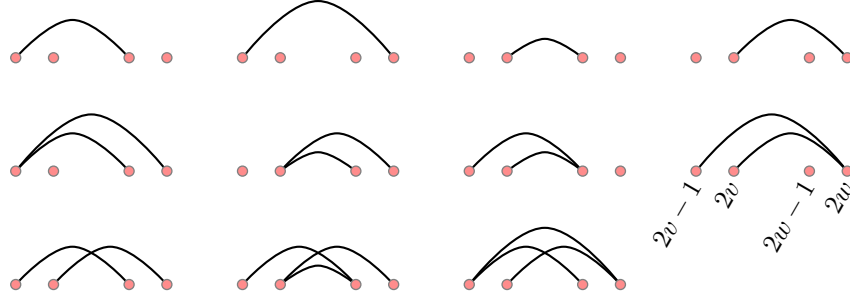
FIGURE 1. The 11 possible ways to add a nonempty set of non-nested edges between $\{2v-1, 2v\}$ and $\{2w-1, 2w\}$.

Lemmata 2 and 4 imply the following.

Corollary 1. *The number of k-queues on n vertices is at most c^{kn} for some absolute constant c.*

It is easily seen that Lemma 1 and Corollary 1 imply a lower bound of $c(\Delta/2-1)\log n$ on the queue-number of some Δ-regular n-vertex graph. To improve this logarithmic bound to polynomial, we now give a more precise analysis of the number of k-queues that also accounts for the number of edges in the graph.

Let $g(n,m)$ be the number of k-queues on n vertices and m edges.

Lemma 5.
$$g(n,m) \leq \begin{cases} \binom{n}{2m} \cdot c^{2m} & \text{, if } m \leq \frac{n}{2} \\ c^n & \text{, if } m > \frac{n}{2}, \end{cases}$$
for some absolute constant c.

Proof. By Lemma 4, we have the upper bound of c^n regardless of m. Suppose that $m \leq \frac{n}{2}$. An m-edge graph has at most $2m$ vertices of nonzero degree. Thus every n-vertex m-edge queue is obtained by first choosing a set S of $2m$ vertices, and then choosing a queue with $|S|$ vertices. The result follows. $\square$

Let $g(n,m,k)$ be the number of k-queues on n vertices and m edges.

Lemma 6. *For all integers k such that $\frac{2m}{n} \leq k \leq m$,*
$$g(n,m,k) \leq \left(\frac{ckn}{m}\right)^{2m}$$
for some absolute constant c.

Proof. Fix nonnegative integers $m_1 \leq m_2 \leq \cdots \leq m_k$ such that $\sum_i m_i = m$. Let $g(n; m_1, m_2, \ldots, m_k)$ be the number of k-queues G on n vertices such that there is a partition $\{E_1, E_2, \ldots, E_k\}$ of $E(G)$, and each $G[E_i]$ is a queue with $|E_i| = m_i$. Then
$$g(n; m_1, m_2, \ldots, m_k) \leq \prod_{i=1}^{k} g(n, m_i).$$

Now $m_1 \leq \frac{n}{2}$, as otherwise $m > \frac{kn}{2} \geq m$. Let j be the maximum index such that $m_j \leq \frac{n}{2}$. By Lemma 6,

$$g(n; m_1, m_2, \ldots, m_k) \leq \left(\prod_{i=1}^{j} \binom{n}{2m_i} c^{2m_i} \right) (c^n)^{k-j}.$$

Now $\sum_{i=1}^{j} m_i \leq m - \frac{1}{2}(k-j)n$. Thus

$$g(n; m_1, m_2, \ldots, m_k) \leq \left(\prod_{i=1}^{j} \binom{n}{2m_i} \right) \left(c^{2m-(k-j)n} \right) \left(c^{(k-j)n} \right)$$

$$\leq c^{2m} \prod_{i=1}^{k} \binom{n}{2m_i}.$$

We can suppose that k divides $2m$. It follows (see [1]) that

$$g(n; m_1, m_2, \ldots, m_k) \leq c^{2m} \binom{n}{2m/k}^k.$$

It is well known [14, Proposition 1.3] that $\binom{n}{t} < (en/t)^t$. Thus with $t = 2m/k$ we have

$$g(n; m_1, m_2, \ldots, m_k) < \left(\frac{cekn}{2m} \right)^{2m}.$$

Clearly

$$g(n, m, k) \leq \sum_{m_1, \ldots, m_k} g(n; m_1, m_2, \ldots, m_k),$$

where the sum is taken over all nonnegative integers $m_1 \leq m_2 \leq \cdots \leq m_k$ such that $\sum_i m_i = m$. The number of such partitions [14, Proposition 1.4] is at most

$$\binom{k+m-1}{m} < \binom{2m}{m} < 2^{2m}.$$

Hence

$$g(n, m, k) \leq 2^{2m} \left(\frac{ce\,kn}{2m} \right)^{2m}.$$

$\square$

Every ordered graph on n vertices is isomorphic to at most $n!$ labelled graphs on n vertices. Thus Lemma 6 has the following corollary.

Corollary 2. *For all integers k such that $\frac{2m}{n} \leq k \leq m$, the number of labelled n-vertex m-edge graphs with queue-number at most k is at most*

$$\left(\frac{ckn}{m} \right)^{2m} n!,$$

for some absolute constant c. $\square$

Proof of Theorem 1. Let k be the minimum integer such that every simple Δ-regular graph with n vertices has queue-number at most k. Thus the number of labelled simple Δ-regular graphs on n vertices is at most the number of labelled graphs with n vertices, $\frac{1}{2}\Delta n$ edges, and queue-number at most k. By Lemma 1 and Corollary 2,

$$\left(\frac{n}{3\Delta}\right)^{\Delta n/2} \leq \left(\frac{ck}{\Delta}\right)^{\Delta n} n! \leq \left(\frac{ck}{\Delta}\right)^{\Delta n} n^n.$$

Hence $k \geq \sqrt{\Delta}n^{1/2-1/\Delta}/(\sqrt{3}c)$. $\qquad\square$

REFERENCES

[1] JÁNOS BARÁT, JIŘÍ MATOUŠEK, AND DAVID R. WOOD. Bounded-degree graphs have arbitrarily large geometric thickness. *Electron. J. Combin.*, 13(1):R3, 2006.

[2] EDWARD A. BENDER AND E. RODNEY CANFIELD. The asymptotic number of labeled graphs with given degree sequences. *J. Combin. Theory Ser. A*, 24:296–307, 1978.

[3] SANDEEP N. BHATT, FAN R. K. CHUNG, F. THOMSON LEIGHTON, AND ARNOLD L. ROSENBERG. Scheduling tree-dags using FIFO queues: A control-memory trade-off. *J. Parallel Distrib. Comput.*, 33:55–68, 1996.

[4] EMILIO DI GIACOMO, GIUSEPPE LIOTTA, AND HENK MEIJER. Computing straight-line 3D grid drawings of graphs in linear volume. *Comput. Geom.*, 32(1):26–58, 2005.

[5] VIDA DUJMOVIĆ, PAT MORIN, AND DAVID R. WOOD. Layout of graphs with bounded tree-width. *SIAM J. Comput.*, 34(3):553–579, 2005.

[6] VIDA DUJMOVIĆ, ATTILA PÓR, AND DAVID R. WOOD. Track layouts of graphs. *Discrete Math. Theor. Comput. Sci.*, 6(2):497–522, 2004.

[7] VIDA DUJMOVIĆ AND DAVID R. WOOD. On linear layouts of graphs. *Discrete Math. Theor. Comput. Sci.*, 6(2):339–358, 2004.

[8] VIDA DUJMOVIĆ AND DAVID R. WOOD. Stacks, queues and tracks: Layouts of graph subdivisions. *Discrete Math. Theor. Comput. Sci.*, 7:155–202, 2005.

[9] SHIMON EVEN AND A. ITAI. Queues, stacks, and graphs. In ZVI KOHAVI AND AZARIA PAZ, eds., *Proc. International Symp. on Theory of Machines and Computations*, pp. 71–86. Academic Press, 1971.

[10] JOSEPH L. GANLEY. Stack and queue layouts of Halin graphs, 1995. Manuscript.

[11] LENWOOD S. HEATH, F. THOMSON LEIGHTON, AND ARNOLD L. ROSENBERG. Comparing queues and stacks as mechanisms for laying out graphs. *SIAM J. Discrete Math.*, 5(3):398–412, 1992.

[12] LENWOOD S. HEATH AND ARNOLD L. ROSENBERG. Laying out graphs using queues. *SIAM J. Comput.*, 21(5):927–958, 1992.

[13] ATSUMI IMAMIYA AND AKIHIRO NOZAKI. Generating and sorting permutations using restricted-deques. *Information Processing in Japan*, 17:80–86, 1977.

[14] STASYS JUKNA. *Extremal combinatorics: with applications in computer science.* Texts in Theoretical Computer Science. Springer, 2001. ISBN 3-540-66313-4.

[15] MARTIN KLAZAR. The Füredi-Hajnal conjecture implies the Stanley-Wilf conjecture. In *Formal power series and algebraic combinatorics (Moscow, 2000)*, pp. 250–255. Springer, Berlin, 2000.

[16] ADAM MARCUS AND GÁBOR TARDOS. Excluded permutation matrices and the Stanley-Wilf conjecture. *J. Combin. Theory Ser. A*, 107(1):153–160, 2004.

[17] BRENDAN D. MCKAY. Asymptotics for symmetric 0-1 matrices with prescribed row sums. *Ars Combin.*, 19(A):15–25, 1985.

[18] EDWARD T. ORDMAN AND WILLIAM SCHMITT. Permutations using stacks and queues. In *Proc. 24th Southeastern International Conf. on Combinatorics, Graph Theory, and Computing*, vol. 96 of *Congr. Numer.*, pp. 57–64. 1993.

[19] SRIRAM V. PEMMARAJU. *Exploring the Powers of Stacks and Queues via Graph Layouts*. Ph.D. thesis, Virginia Polytechnic Institute and State University, U.S.A., 1992.

[20] VAUGHAN R. PRATT. Computing permutations with double-ended queues. Parallel stacks and parallel queues. In *Proc. 5th Annual ACM Symp. on Theory of Computing (STOC '73)*, pp. 268–277. ACM, 1973.

[21] S. RENGARAJAN AND C. E. VENI MADHAVAN. Stack and queue number of 2-trees. In DING-ZHU DU AND MING LI, eds., *Proc. 1st Annual International Conf. on Computing and Combinatorics (COCOON '95)*, vol. 959 of *Lecture Notes in Comput. Sci.*, pp. 203–212. Springer, 1995.

[22] FARHAD SHAHROKHI AND WEIPING SHI. On crossing sets, disjoint sets, and pagenumber. *J. Algorithms*, 34(1):40–53, 2000.

[23] GÁBOR TARDOS. On 0-1 matrices and small excluded submatrices. *J. Combin. Theory Ser. A*, 111(2):266–288, 2005.

[24] ROBERT E. TARJAN. Sorting using networks of queues and stacks. *J. Assoc. Comput. Mach.*, 19:341–346, 1972.

[25] DAVID R. WOOD. Queue layouts of graph products and powers. *Discrete Math. Theor. Comput. Sci.*, 7(1):255–268, 2005.

[26] NICHOLAS WORMALD. *Some problems in the enumeration of labelled graphs*. Ph.D. thesis, Newcastle-upon-Tyne, United Kingdom, 1978.

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